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Motion of a Rigid Body in Body-Fixed Coordinate System — an Example for Autoparrallel Trajectories in Spaces with Torsion

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Abstract

We use a recently developed action principle in spaces with curvature and torsion to derive the Euler equations of motion for a rigid body within the body-fixed coordinate system. This serves as an example that the particle trajectories in a space with curvature and torsion follow the straightest paths (autoparallels), not the shortest paths (geodesics), as commonly believed.

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1) Introduction

Since their discovery more than two centuries ago, the Euler equations

$$\dot{\mathbf{L}} + \boldsymbol{\Omega} \times \mathbf{L} = \mathbf{M} \tag{1}$$

have played a key role in understanding the rotations of a rigid body around a fixed point O. (see for example [1] – [5] and the references therein). The vectors refer to the body-fixed system, $\mathbf{L}$ being the angular momentum, $\boldsymbol{\Omega}$ the instantaneous angular velocity, and $\mathbf{M}$ the moments of the external forces. The dot denotes differentiation with respect to the time t. The unit basis vectors $\varepsilon_i (i = 1, 2, 3)$ in B may be assumed to point along the eigenvectors of the inertia tensor with respect to O. Then the angular moment $\mathbf{L}$ has the components $L_i = I_i \Omega^i$ (no sum over i).

To describe also the translational motion of the rigid body one usually chooses the point O to coincide with the center of mass moving through space. The motion satisfies the additional equations:

$$\dot{\mathbf{P}} + \boldsymbol{\Omega} \times \mathbf{P} = \mathbf{F} \tag{2}$$

where $\mathbf{P} = M\mathbf{V}$ is the linear momentum, M the body's total mass, $\mathbf{V}$ the velocity of the center of mass, and $\mathbf{F}$ the driving force. Presently, the equations (1) and (2) play an important role in missile dynamics analysis [5].

Usually, the two sets of equations are derived as the Newton equations of motion in the *stationary* reference system S:

$$\dot{\mathbf{l}} = \mathbf{m}, \tag{3}$$

$$\dot{\mathbf{p}} = \mathbf{f}. \tag{4}$$

Following general conventions, we distinguish vectors and tensors in S and B by using for the small letters with Greek indexes and the capital letters with Latin indexes, respectively.

An transformation

$$\Omega^i = e^i{}_\mu \omega^\mu, \tag{5}$$

$$V^i = \varepsilon^i{}_\mu v^\mu, \tag{6}$$

carries the of the angular and linear velocities ω^μ and v^μ to the moving body-fixed system B (see [1] – [3]).

The 3×3 matrix elements $e^i{}_\mu$ and $\varepsilon^i{}_\mu$ depend on the coordinates q^μ in the system S. They satisfy $\partial_{[\nu} e^i{}_{\mu]} \neq 0$, $\partial_{[\nu} \varepsilon^i{}_{\mu]} \neq 0$, making the transformation (5) and (6) *anholonomic coordinate transformations* (see [1]– [7] for the general theory of the dynamics in anholonomic coordinates is presented).

Under the transformation (5), the equations (3) go over into (1) and the additional term $(\mathbf{\Omega} \times \mathbf{L})^k = \epsilon^k{}_{ij} \Omega^i L^j$ arises as the moment of the gyroscopic force. The additional term $(\mathbf{\Omega} \times \mathbf{P})^k = \epsilon^k{}_{ij} \Omega^i P^j$ in equation (2) arise similarly. Both terms are a consequence of the anholonomy of the transformation (5), (6).

Within the stationary system S, Hamilton's action principle serves to derive the equations of motion (3) and (4). If one transforms the classical action to the system B, however, the description involves nonholonomic coordinates and a naive application of Hamilton's principle produces wrong equations of motion lacking the additional gyroscopic moments. In 1901, Poincaré showed [6] how to vary an action expressed in terms of nonholonomic coordinates. Following his treatment, one certainly recovers the gyroscopic forces. His treatment is reviewed in the Textbook [7].

The purpose of this note is to point out that Poincaré's treatment may be viewed geometrically as an application of a recently proposed action principle in spaces with curvature and torsion to the spinning top within the body-fixed reference system. The motivation for such a consideration derives from the fact that, in the literature on gravity with curvature and torsion [8] (for the geometry of such spaces see [9]), there is a widespread belief that in spaces with torsion, spinless particles move on shortest paths. However, it was discovered in Ref. [10] (when solving the path integral of the hydrogen atom) that the correct trajectories are the straightest paths in a given geometry. In Ref. [11], a classical action principle was found to comply with this physical fact.

The key to the new action principle is the observation, that a space M_q with nonzero Riemann curvature and torsion may be mapped locally into a euclidean space M_x by an

anholonomic transformation $\dot{x}^i = e^i{}_\mu \dot{q}^\mu$ [12], [10]. For the particle with mass m in M_x, the equations of motion $\ddot{x}^i = 0$ yield straight-line trajectories. Under an anholonomic transformation, these lines go over into autoparallels satisfying the equation $\ddot{q}^\mu + \Gamma_{\alpha\beta}{}^\mu \dot{q}^\alpha \dot{q}^\beta = 0$ with $\Gamma_{\alpha\beta}{}^\mu = e_i{}^\mu \partial_\alpha e^i{}_\beta$ being the Cartan connection.

The orbits in the euclidean space M_x may be derived from Hamilton's principle $\delta A_x = 0$ applied to the classical action $A_x = \int_{t_1}^{t_2} dt \frac{1}{2} m \dot{x}^i \dot{x}^i$. Under the anholonomic transformation $\dot{x}^i = e^i{}_\mu \dot{q}^\mu$, this action goes into the curvilinear form $A_q = \int_{t_1}^{t_2} dt \frac{1}{2} m g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu$. In the space M_q, a naive application of Hamilton's principle $\delta A_q = 0$ produces wrong equations of motion in the space $\mathrm{M_q}$. One finds $\ddot{q}^\mu + \bar{\Gamma}_{\alpha\beta}{}^\mu \dot{q}^\alpha \dot{q}^\beta = 0$ which are the equations for the geodesics rather than the autoparallels; they lack torsion force $2m S_\alpha{}^\mu{}_\beta \dot{q}^\alpha \dot{q}^\beta$.

The problem of describing the dynamics of a rigid body within the body-fixed frame may be formulated is somewhat analogous. Here the parameter space of Euler angles has a Riemann curvature. By going to the body-fixed frame and describing the system in terms of non-holonomic coordinates, the space becomes affine-flat but possesses torsion. The affine connection in the space of anholonomic coordinates. is obtained from the spatial derivatives of the transformation matrices in (5) and (6). The associated Cartan curvature tensor $R_{\mu\nu\lambda}{}^\kappa \equiv e_i{}^\kappa (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^i_\lambda$ vanishes, making the connection affine-flat. The antisymmetric part of the connection $S_{\alpha\beta}{}^\gamma = \frac{1}{2}(\Gamma_{\alpha\beta}{}^\gamma - \Gamma_{\beta\alpha}{}^\gamma)$, which is a tensor, is nonzero giving rise to a nonvanishing Riemann curvature tensor $\bar{R}_{\mu\nu\lambda}{}^\kappa \neq 0$. The latter is formed from the Levi-Cevita connection, also called Christoffel symbol $\bar{\Gamma}_{\mu\nu\lambda} = \frac{1}{2}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)$, where $g_{\mu\nu} = e^i{}_\mu e^i{}_\nu$ is the Riemann metric in the space of anholonomic coordinates.

We shall apply the variational principle of Ref. [11] and derive, within the body-fixed reference system B, both the Euler equations (1) and equations for the translational motion (2).

2) The SO(3) geometry in a rotating body-fixed system produced by an anholonomic transformation from the rest system

Consider first a rigid body rotating around a fixed point. Within the body-fixed sys-

tem B, we introduce anholonomic coordinates Φ^i corresponding to the transformation (5). They are the components of the body's rotation vector in the axis-angle parametrization. The anholonomic transformation (5) defines their infinitesimal increments $d\!\!{}^-\Phi^i = \Omega^i dt$. For a precise specification, let us go to the system S where the standard Euler angles $\alpha = \varphi^3, \beta = \varphi^2, \ \gamma = \varphi^1$ [10] parametrize (holonomically) the body's configuration space $\mathrm{M}^{(3)} = \mathrm{SO}(3)$. The components of the angular velocity in the system B are $\Omega^1 = s_1\dot{\varphi}^2 - c_1 s_2 \dot{\varphi}^3, \ \Omega^2 = c_1\dot{\varphi}^2 + s_1 s_2\dot{\varphi}^3, \Omega^3 = \dot{\varphi}^1 + c_2\dot{\varphi}^3$ (with the short notation $c_\mu = \cos\varphi^\mu, \ s_\mu = \sin\varphi^\mu$). The basic relation

$$d\!\!{}^-\Phi^i = e^i{}_\mu(\varphi) d\varphi^\mu \tag{7}$$

has coefficients $e^i{}_\mu$ which form the matrix

$$e = \begin{pmatrix} 0 & s_1 & -c_1 s_2 \\ 0 & c_1 & s_1 s_2 \\ 1 & 0 & c_2 \end{pmatrix}. \tag{8}$$

The symbol $d\!\!{}^-$ stands for increments do not belong to an integrable function; they do not satisfy the Schwarz integrability condition which would read, with the coefficients of (7), $\partial_\mu e^i{}_\nu - \partial_\nu e^i{}_\mu = 0$.

The metric $g = (g_{\alpha\beta})$ and its inverse $g^{-1} = \left(g^{\alpha\beta}\right)$ have the matrices

$$g = \begin{pmatrix} 1 & 0 & c_2 \\ 0 & 1 & 0 \\ c_2 & 0 & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 1 & 0 & -c_2 \\ 0 & {s_2}^2 & 0 \\ -c_2 & 0 & 1 \end{pmatrix}, \tag{9}$$

where $|g| = \det g = {s_2}^2; \quad g^{-1} = |g|^{-1}$ is the determinant of g. From g we calculate the Christoffel symbol:

$$\bar{\Gamma}_{\mu\nu\lambda} = -\frac{\sqrt{|g|}}{2}\left(\delta_{1\mu}\delta_{2\nu}\delta_{3\lambda} + \delta_{1\nu}\delta_{2\mu}\delta_{3\lambda} + \delta_{1\lambda}\delta_{2\mu}\delta_{3\nu} + \delta_{1\lambda}\delta_{2\nu}\delta_{3\mu} - \delta_{1\mu}\delta_{2\lambda}\delta_{3\nu} - \delta_{1\nu}\delta_{2\lambda}\delta_{3\mu}\right).$$

The associated Riemannian Ricci tensor is $\bar{R}_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta}$ with a constant nonzero scalar Riemannian curvature $\bar{R} = 3/2$. The only nonzero components of the affine-flat Cartan connection are

$$\Gamma_{12}{}^{1} = \quad \Gamma_{23}{}^{3} = \frac{c_2}{s_2}, \quad \Gamma_{12}{}^{3} = \quad \Gamma_{23}{}^{1} = -\frac{1}{s_2}, \quad \Gamma_{13}{}^{2} = s_2. \tag{10}$$

From these we find the antisymmetric part is the torsion tensor $S_{\alpha\beta\gamma} = -\frac{1}{2}\sqrt{|g|}\epsilon_{\alpha\beta\gamma}$. A transforming to the local basis ε^i yields the so-called object of anholonomy [1]– [4]

$$2S_{ijk} = 2e_i^\alpha e_j^\beta e_k^\mu S_{\alpha\beta\gamma} \tag{11}$$

whose explicit form is

$$2S_{ijk} = -\epsilon_{ijk}, \tag{12}$$

where ϵ_{ijk} denotes the Levi-Civita antisymmetric tensor in Cartesian coordinates ($\epsilon_{123} = 1$) [9]. This geometry prepares the ground for the action principle to be developed.

3) Action principle for the rotational motion in the body-fixed system B

Pure rotations around an arbitrary fixed point of the body are governed by the Euler equations (1). The kinetic energy is $T = \frac{1}{2} I_i \Omega^i \Omega^i$. In S, the Lagrangian is $L_S = \frac{1}{2} g^{\rm kin}_{\mu\nu} \dot{\varphi}^\mu \dot{\varphi}^\nu - U(\varphi)$ with U being the potential energy and $g^{\rm kin}_{\mu\nu} = I_i e^i{}_\mu e^i{}_\nu$ the "kinetic-energy-metric". The latter metric differs from geometric metric $g_{\mu\nu}$ of Eq. (9).[1] In this note, the kinetic-energy-metric $g^{\rm kin}_{\mu\nu}$ [10] will play no role. The potential term $U(\varphi)$ is irrelevant for our considerations and will be dropped.

An application of Hamilton's action principle $\delta A_S = 0$ to the classical action $A_S = \int_{t_1}^{t_2} L_S dt$ in the system S certainly yields the correct equations of motion. Under the transformation (5), they become the Euler equations (1).

Let us now transform the action A_S via (5) to the body-fixed system B. The result is very simple:

$$A_B = \int_{t_1}^{t_2} dt L_B = \int_{t_1}^{t_2} dt \frac{1}{2} I_i \dot{\Phi}^i \dot{\Phi}^i. \tag{13}$$

By applying naively Hamilton's action principle we would find the equations $I_i \ddot{\Phi}^i = I_i \dot{\Omega}^i = 0$ for each i which are *not* the correct Euler equations (1) — the gyroscopic moments being missed. The contradiction is caused by the fact that the variations $\delta\varphi^\mu(t)$ in the space of Euler angles M_φ and the variations $\delta\Phi^i(t)$ in the space of anholonomic coordinates M_Φ, are related with each other in a path dependent way ("nonlocal" on the time axis) [11]. Explicitly, there is the functional equation

$$\Phi^i_2 = \Phi^i_1 + \int_{t_1}^{t_2} e^i{}_\mu[\varphi(t)] \dot{\varphi}^\mu(t) dt. \tag{14}$$

Its most important consequence is that closed paths in the space M_φ are not mapped into closed paths in the space M_Φ, due to a nonvanishing Burgers vector $b^i = \oint e^i{}_\mu d\varphi^\mu \neq 0$.

The usual Hamilton action principle in the system S proceeds by considering a variation $\delta\varphi^\mu(t) = \bar{\varphi}^\mu(t) - \varphi^\mu(t)$ between two paths $\bar{\varphi}^\mu(t)$, $\varphi^\mu(t)$ ($t \in [t_1, t_2]$) with common ends:

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[1] Note that Einstein's summation convention is extended to include also diagonal components of the inertia tensor.

$\bar{\varphi}^\mu(t_{1,2}) = \varphi^\mu(t_{1,2})$. Together, they form a *closed* path in the space M_φ. The above naive application of Hamilton's action principle in the system B employed analogous closed-path variations $\delta\Phi^i(t) = \bar{\Phi}^i(t) - \Phi^i(t)$ between two paths $\bar{\Phi}^i(t), \Phi^i(t)$ $(t \in [t_1, t_2])$, with also common ends: $\bar{\Phi}^i(t_{1,2}) = \Phi^i(t_{1,2})$. This, however, cannot be correct. Under a transformation (5), variations $\delta\varphi^\mu$ do *not* produce closed-path variations $\delta\Phi^i$ in the space M_Φ. The nonzero Burgers vector $b^i \neq 0$ of the anholonomy causes a closure failure. This has to be accounted for in a correct derivation of the equations of motion.

The anholonomy of the transformation (5) requires distinguishing two types of variations of the paths in the space M_Φ: closed path variations $\delta\Phi^i(t)$ of the tpye described above, and *anholonomic* variations $\bar{\delta}\Phi^i(t)$, which are images of the $\delta\varphi^\mu(t)$ variations in the space M_φ. Only the latter will produce the correct equations of motion from an action principle in the system B.

To calculate the variation of the classical action in the system B we derive the following simple formula for the anholonomic variation $\bar{\delta}\dot{\Phi}^i$ of the angular velocity $\Omega^i = \dot{\Phi}^i$:

$$\bar{\delta}\dot{\Phi}^i = \frac{d}{dt}\delta\Phi^i + \epsilon^i{}_{jk}\dot{\Phi}^j\delta\Phi^k \tag{15}$$

(corresponding to formula (9) in Ref. [11]).

Indeed, the transformation (5) and the definition (7) lead to the equation

$$\bar{\delta}\dot{\Phi}^i = \delta(e^i{}_\mu\dot{\varphi}^\mu) = e^i{}_\mu\delta\dot{\varphi}^\mu + \partial_\nu e^i{}_\mu\dot{\varphi}^\mu\delta\varphi^\nu = \frac{d}{dt}\left(e^i{}_\mu\delta\varphi^\mu\right) + \left(\partial_\mu e^i{}_\nu - \partial_\nu e^i{}_\mu\right)\dot{\varphi}^\nu\delta\varphi^\mu \tag{16}$$

which by (11) becomes

$$\bar{\delta}\Phi^i = \frac{d}{dt}\delta\Phi^i + 2S_{kj}{}^i\dot{\Phi}^j\delta\Phi^k \tag{17}$$

leading to (15) via (12). The variations $\delta\varphi^\mu$ are still performed in the system S. They can be mapped *locally* into holonomic closed-path variations $\delta\Phi^i$ in the system B by $\delta\Phi^i = e^i{}_\mu\delta\varphi^\mu$. They have the fixed-endpoint property $\delta\Phi^i(t_{1,2}) = 0$, reflecting the closed-path condition $\delta\varphi^\mu(t_{1,2}) = 0$ in the system S.

Let us emphasize that our final nonholonomic variations (15) are completely intrinsic to the system B. Applying these to the action A_B, an integration by parts with the boundary conditions $\delta\Phi^i(t_{1,2}) = 0$ leads to

$$\bar{\delta} A_B = \int_{t_1}^{t_2} dt I_i \dot{\Phi}^i \bar{\delta}\Phi^i = -\int_{t_1}^{t_2} dt \delta\Phi^i (\dot{\mathbf{L}} + \mathbf{\Omega} \times \mathbf{L})^i \tag{18}$$

Since $\delta\Phi^i$ are now ordinary (holonomic) variations, we find the Euler equations (1). Thus the anholonomic action principle

$$\bar{\delta} A_B = 0 \tag{19}$$

produces the correct equations of motion in the body-fixed system B.

4) Action principle for general motion in the body-fixed system B

To extend the action principle to the general motion of a rigid body one usually chooses the point O in B to be the center of mass of the body and studies the movements of O and rotations around O. Then I_j are the body's principal moments of inertia. The configuration space is a 6-dimensional manifold $\mathrm{M}^{(6)} = R^{(3)} \times \mathrm{SO}(3)$ and the kinetic energy $T = T_{\mathrm{trans}} + T_{\mathrm{rot}}$ consists of two terms: The first is the translational kinetic energy $T_{\mathrm{trans}} = \frac{1}{2} M V^i V^i$ in the system B the second is the rotational kinetic energy $T_{\mathrm{rot}} = \frac{1}{2} I_i \Omega^i \Omega^i$ in the system B.

In the system S, the position and orientation of the body are parametrized by the (holonomic) coordinates $\{x^\mu, \varphi^\mu\}_{\mu=1,2,3} = \left\{q^M\right\}_{M=1,2,\ldots 6}$, with x^μ being the mass center coordinates and φ^μ the Euler angles.

The body-fixed basis vectors in B (see Fig. 1) are some function of the Euler angles $\varepsilon_i(\varphi)$. Their components in the cartesian basis of the system S are $\varepsilon_i{}^\mu(\varphi)$. They form a 3×3 orthogonal matrix $\varepsilon(\varphi) = (\varepsilon_i{}^\mu)$, $\varepsilon^{-1} = (\varepsilon^i{}_\mu) = \varepsilon^T$. The components of the velocity of the center of mass in the system S are $v^\mu = \dot{x}^\mu$. The components of $\dot{x}^\mu$ with respect to the basis ε_i are $V^i = \varepsilon^i{}_\mu \dot{x}^\mu$.

The last relation permits us to introduce a new set of anholonomic coordinates X^i describing the center of mass motion in the system S as seen from the system B. By analogy with formula (7), we define the infinitesimal increments $\bar{d}X^i$ as:

$$đX^i = \varepsilon^i{}_\mu(\varphi)dx^\mu. \tag{20}$$

The set $\{X^i, \Phi^i\}_{i=1,2,3} = \left\{Q^I\right\}_{I=1,\dots,6}$ gives a complete set of an anholonomic coordinates for the configuration space $\mathrm{R}^{(3)} \times \mathrm{SO}(3)$ of the moving body.

Let us write down the anholonomic geometry of this space. Introducing 6×6 matrices, whose rows and columns are labeled by capital Latin and Greek letters,

$$\mathcal{E} = \begin{pmatrix} e & 0 \\ 0 & \varepsilon \end{pmatrix} = \left(\mathcal{E}^I{}_\Lambda\right), \quad \mathcal{G} = (\mathcal{G}_{\Lambda\Sigma}) = \mathcal{E}^T\mathcal{E} = \begin{pmatrix} g & 0 \\ 0 & \hat{1} \end{pmatrix}, \dots \tag{21}$$

the geometry possesses the affine connection $\Gamma_{\Lambda\Sigma}{}^\Delta = \mathcal{E}_I{}^\Delta\partial_\Lambda\mathcal{E}^I{}_\Sigma$ with the torsion tensor $S_{\Lambda\Sigma}{}^\Delta = \Gamma_{[\Lambda\Sigma]}{}^\Delta \neq 0$ and a vanishing Cartan curvature tensor $R_{\Lambda\Sigma,\Delta}{}^\Xi = 0$, implying an nonvanishing Riemann curvature tensor $\bar{R}_{\Lambda\Sigma,\Delta}{}^\Xi \neq 0$. This geometry is a simple extension of the 3-dimensional anholonomic geometry on SO(3) described in Section 2. It reflects the structure of the present configuration space $R^{(3)} \times \mathrm{SO}(3)$, i.e., the space of configurations of the rigid body comprising translations of the center of mass and rotations around it.

As a direct consequence of the definition (20), we have the relation $V^i = \varepsilon^i{}_\mu\dot{x}^\mu = \dot{X}^i$ which, together with (7), leads to the following simple form of the rigid body's Lagrangian in the system B:

$$L_B = \frac{M}{2}\dot{X}^i\dot{X}^i + \frac{1}{2}I_i\dot{\Phi}^i\dot{\Phi}^i. \tag{22}$$

As before, a naive application of Hamiltons principle in the system B would produce a wrong equation $M\ddot{X}^i = 0$ for the motion of the center-of-mass. The correct anholonomic variation of the velocity

$$\eth\dot{X}^i = \frac{d}{dt}(\delta X^i) + \varepsilon^i{}_{jk}\left(\delta\Phi^j\dot{X}^k - \dot{\Phi}^j\delta X^k\right). \tag{23}$$

Indeed, the transformation (6) and the definition (20) lead to

$$\eth\dot{X}^i = \delta(\varepsilon^i{}_\mu\dot{x}^\mu) = \frac{d}{dt}(\epsilon^i{}_\mu\delta x^\mu) + \partial_\lambda\epsilon^i{}_\mu(\dot{x}^\mu\delta\varphi^\lambda - \dot{\varphi}^\lambda\delta x^\mu). \tag{24}$$

The variations δx^μ may be mapped *locally* into holonomic closed-path variations $\delta X^i = \varepsilon^i{}_\mu\delta x^\mu$ in the system B with the property $\delta X^i(t_{1,2}) = 0$ reflecting the closed path condition $\delta x^\mu(t_{1,2}) = 0$. Formula (23) follows using the equations $\dot{x}^\mu = \varepsilon^\mu{}_k\dot{X}^k$, $\delta x^\mu =$

$\varepsilon^\mu{}_k\ \delta X^k \quad (\varepsilon^\mu{}_k \varepsilon^k{}_\nu = \delta^\mu_\nu)$, $\dot\varphi^\lambda = e^\lambda{}_j \dot\Phi^j, \quad \delta\varphi^\lambda = e^\lambda{}_j \delta\Phi^j \quad (e^\lambda{}_j e^j{}_\nu = \delta^\lambda_\nu)_j$, and the relation $(d\varepsilon\varepsilon^T)_{ij} = -\varepsilon_{ijk} d\Phi^k$.

By applying the anholonomic variation (23) to the translational energy, we obtain

$$\bar\delta T_{\text{trans}} = \frac{d}{dt}(M\dot X^i \delta X^i) - M(\ddot X^i + \varepsilon^i{}_{jk}\dot\Phi^j \dot X^k)\delta X^i. \tag{25}$$

An integration by parts using of the fixed-ends conditions $\delta X^\mu(t_{1,2}) = 0$ leads to the following expression for the total anholonomic variation of the action $A_B = \int_{t_1}^{t_2} L_B dt$:

$$\bar\delta A_B = \int_{t_1}^{t_2} dt (I_i \dot\Phi^i \bar\delta\dot\Phi + M\dot X^i \bar\delta \dot X^i) = \tag{26}$$

$$= -\int_{t_1}^{t_2} dt [\delta\Phi^i (\mathbf{L} + \mathbf{\Omega}\times\mathbf{L})^i + \delta X^i(\dot{\mathbf{P}} + \mathbf{\Omega}\times\mathbf{P})^i] \tag{27}$$

From the second term, we find the equations (2) for the translational motion.

5) **Conclusion**

By subjecting the action A_B in the body-fixed system to the new anholonomic variations with respect to translational and rotational degrees of freedom of the rigid body we have been able to derive both the correct Euler equations (1) and the equations (2) completely within the body-fixed system without reference to the stationary systems.

The existence of such an action principle intrinsic to the body-fixed system may not be only of aesthetic value but may also have important practical consequences.

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